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BEHAVIOR OF DERIVATIVES OF EIGENVALUES AND EIGENVECTORS IN CURVE VEERING AND MODE LOCALIZATION AND THEIR RELATION TO CLOSE EIGENVALUES

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The problem to measure the phenomena of eigenvalue curve veering and mode localization is addressed in this paper. The second derivative of an eigenvalue and the first derivative of an eigenvector are taken as the measures, numerically showing curve veering and mode localization. Based on the measurement, close eigenvalues, as a key factor for the occurrence of the phenomena, are defined. Two eigenvalues are considered to be close, if their difference is small enough to cause the occurrence of the phenomena. The curve veering and mode localization can be noted by comparison of the derivatives with a critical value and hence the associated eigenvalues are close. Weakly coupled springs are given as an example.

1. INTRODUCTION

Mode localization and eigenvalue curve veering are the phenomena of rapid and even violent changes in dynamic modes. Earlier studies showed that mode localization can be observed in linear periodic systems [1–7]. Non-linear localized modes were further found in periodic oscillators [8, 9] and continuous systems [10, 11]. Rapid change of eigenvalues, known as curve veering or loci veering, has been noted in various structural systems, such as a curved beam [12], rectangular membrane [13], orthotropic rectangular plates [14], cables and chains [15], rotating circular string [16], coupled oscillators [17], multi-span beams [18], coupled pendulums [4, 19], blade assemblies [5], and space structures [3, 7] as well.

The occurrence of curve veering and mode localization suggests that the dynamic system is very sensitive to a parameter. Attention must be paid to the significance of the sensitivity, for it affects the dynamic modes greatly. A dynamic model can be far from the assumed prototype, caused by a small variation, such as a manufacturing error, a geometrical irregularity, or a mistuned parameter. For these reasons, the subject is worthy of both a theoretical study and a guide for engineering practice.

It is proved that the existence of close eigenvalues, or near frequencies, in a dynamic system is likely to cause the occurrence of curve veering and mode localization [13–19]. When some disorder is introduced in nearly periodic structures with weak internal coupling, both strong mode localization and veering of the eigenvalue occur, indicating that these are two manifestations of the same drastic phenomenon [19]. Therefore, close eigenvalues are a precondition of the occurrence in these structures. Moreover, the closeness

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of the eigenvalues also determines the degree of the phenomena. The study of veering and localization, when they occur and how they are measured, is an interesting topic. In other words, a numerical measure of the phenomena must be studied. Based upon the measurement, how two close eigenvalues can lead to the occurrence of the phenomena is of interest.

Leissa [13] suggested using the second difference of eigenvalues to measure curve veering and its application was illustrated using a rectangular membrane. It is known that the derivatives of a function behave more violently than the function itself. Therefore, the derivatives of eigenvalues and eigenvectors, as the rate of their change, can be measures for the phenomena of veering and localization.

As an example of curve veering and mode localization with close eigenvalues, weakly coupled springs [17] are considered in the paper. The occurrence of the phenomena and the behavior of the derivatives of eigenvalues and eigenvectors will be examined. The relation of the derivatives to two close eigenvalues will be set up later.

2. EXAMPLE OF CURVE VEERING AND MODE LOCALIZATION

Consider two springs with stiffness k_1 and k_2 and masses m_1 and m_2 connected by a weak spring k, as shown in Figure 1. The vibration of the springs is along the axis X. The eigenvalue equation of the springs is such that

$$\left(\begin{bmatrix} k_1 + k & -k \\ -k & k_2 + k \end{bmatrix} - \lambda \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\right)\begin{Bmatrix} \varphi_1 \\ \varphi_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \tag{1}$$

where λ is the eigenvalue and $\boldsymbol{\varphi}$ is the eigenvector, which is normalized by

$$m_1(\varphi_1)^2 + m_2(\varphi_2)^2 = 1. \tag{2}$$

There are two modes, in which the eigenvalues can be directly written out as

$$\lambda_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}, \tag{3}$$

$$\lambda_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \tag{4}$$

where

$$a = m_1 m_2, \qquad b = -(k_1 + k)m_2 - (k_2 + k)m_1, \tag{5, 6}$$

$$c = k_1 k + k_1 k_2 + k_2 k. \tag{7}$$

The normalized eigenvectors can be solved from equations (1) and (2) as

$$\begin{Bmatrix} \varphi_1 \\ \varphi_2 \end{Bmatrix}_i = \begin{Bmatrix} \dfrac{1}{\sqrt{m_1 + m_2 s_i^2}} \\ \dfrac{s_i}{\sqrt{m_1 + m_2 s_i^2}} \end{Bmatrix}, \quad i = 1, 2, \tag{8}$$

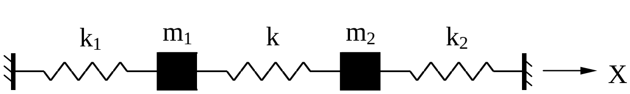

Figure 1. Weakly coupled springs.

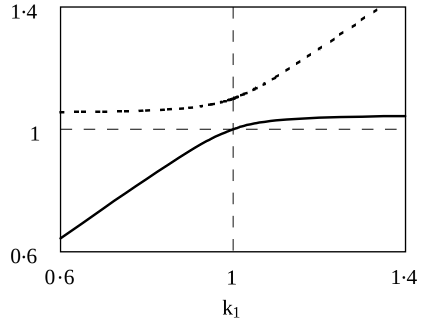

Figure 2. Eigenvalues versus k_1 ($k_2 = 1$, $k = 0{\cdot}05$, $m_1 = m_2 = 1$). —— λ_1, ····· λ_2.

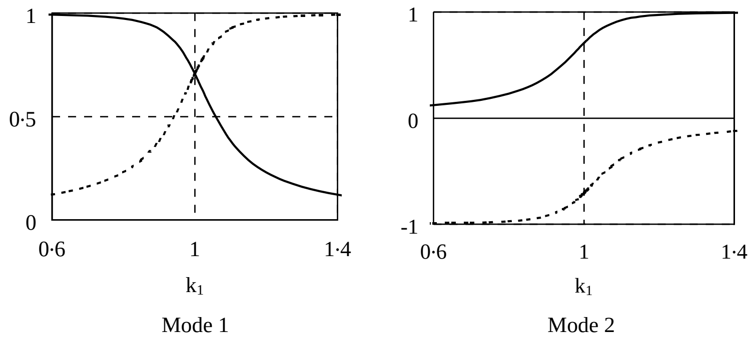

Figure 3. Eigenvectors versus k_1 ($k_2 = 1$, $k = 0{\cdot}05$, $m_1 = m_2 = 1$). —— φ_1, ····· φ_2.

where

$$s_i = \frac{k_1 + k - \lambda_i m_1}{k}. \tag{9}$$

To show close eigenvalues and the phenomena of veering and localization, two cases are considered, with k_1 taken as a variable and different k.

Case I (k = 0·05)

Given $k_2 = 1$, $m_1 = m_2 = 1$, and $k = 0{\cdot}05$, the eigenvalues and eigenvectors of two modes following various values of k_1 are shown in Figures 2 and 3.

Case II (k = 0·01)

Case II is considered with the same stiffness and mass as Case I except $k = 0{\cdot}01$. The change of eigenvalues and eigenvectors versus k_1 is shown in Figures 4 and 5.

The occurrence of curve veering and mode localization is observed in both cases. By comparison of the figures, the following conclusions can be drawn.

(1) In the two cases with different values for k, two eigenvalues are close to each other at $k_1 \approx k_2 = 1$. The eigenvalues are much closer in the latter case for $k = 0{\cdot}01$, which is smaller than 0·05 of the first case. Therefore, a smaller k produces a closer pair of eigenvalues.

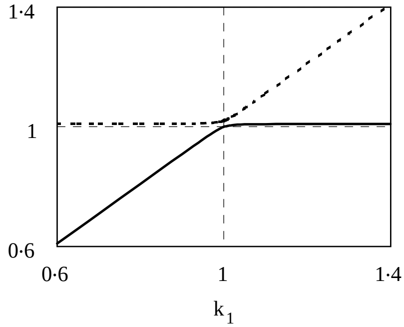

Figure 4. Eigenvalues versus k_1 ($k_2 = 1$, $k = 0{\cdot}01$, $m_1 = m_2 = 1$). —— λ_1, ····· λ_2.

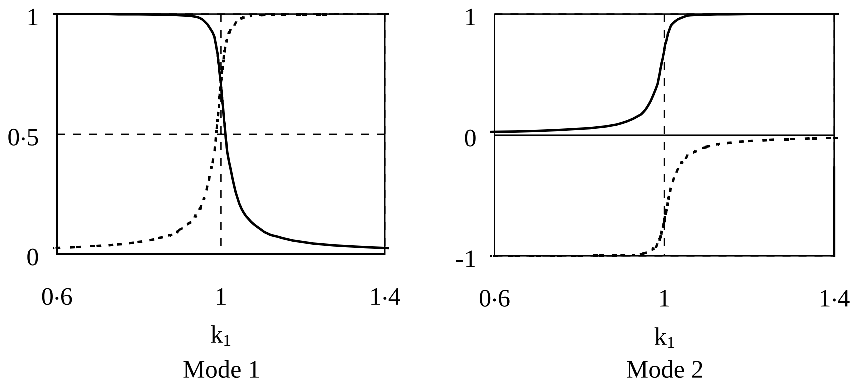

Figure 5. Eigenvectors versus k_1 ($k_2 = 1$, $k = 0{\cdot}01$, $m_1 = m_2 = 1$). —— φ_1, ····· φ_2.

(2) A rapid change of eigenvalues, showing the occurrence of the eigenvalue curve veering, is observed in both cases. The more rapid one comes from closer eigenvalues. The change happens in a range, which is small when two eigenvalues are getting close. As two eigenvalues become close, their curves repel each other, suggesting a violent veering.
(3) The mode localization occurs with close eigenvalues. The change of modal shapes is in a narrow range of the stiffness. The range is smaller as the eigenvalues become closer. Therefore, a close pair of eigenvalues brings the occurrence of a more rapid localization in a smaller range of k_1.

In summary, the value of stiffness k determines the degree of closeness of the eigenvalues. A proper ratio of the stiffness of the two springs, which is near one, is another factor for the occurrence of the phenomena, besides the eigenvalues being close. Much closer eigenvalues from a smaller k lead to a more rapid veering and localization in a smaller range of variation of k_1.

Although the occurrence of veering and localization is given with the variation of the stiffness k_1, the phenomena can also be observed from a variation of the other stiffness or masses, i.e., any of k_2, m_1, and m_2 in the spring–mass system. Curve veering may not exist between the modes of repeated eigenvalue. This can be demonstrated by letting $k_1 = k_2$, $m_1 = m_2$, and k varied in the weakly coupled springs, yielding two lines crossing at a point with a repeated eigenvalue. Petyt and Fleischer [12] showed that the frequencies for the "odd" modes, which are symmetric about the curved beam, can smoothly cross those of the "even" antisymmetric modes.

The curve veering and mode localization in Case II are proved to be more violent than Case I by comparison of the figures. In order to have a precise measure for the occurrence and the degree of the phenomena, a numerical description is required.

3. DERIVATIVES OF EIGENVALUE AND EIGENVECTOR

The partial derivatives of the eigenvalues and eigenvectors for the weakly coupled springs will be examined in this section.

3.1. DERIVATIVES WITH RESPECT TO STIFFNESS k_1

From equations (3) and (4), the first partial derivatives of the eigenvalues with respect to k_1 can be obtained as

$$\frac{\partial \lambda_1}{\partial k_1} = \frac{1}{2m_1}\left[1 - \frac{(k_1 + k)m_2 - (k_2 + k)m_1}{\sqrt{b^2 - 4ac}}\right], \tag{10}$$

$$\frac{\partial \lambda_2}{\partial k_1} = \frac{1}{2m_1}\left[1 + \frac{(k_1 + k)m_2 - (k_2 + k)m_1}{\sqrt{b^2 - 4ac}}\right]. \tag{11}$$

The derivatives versus the stiffness k_1, with three cases of 0·05, 0·01, and 0·005 for k, and $k_2 = 1$, are shown in Figure 6.

The second derivatives of the eigenvalues are given as

$$\frac{\partial^2 \lambda_1}{\partial k_1^2} = -\frac{2k^2 m_2^2}{\sqrt{(b^2 - 4ac)^3}}, \qquad \frac{\partial^2 \lambda_2}{\partial k_1^2} = \frac{2k^2 m_2^2}{\sqrt{(b^2 - 4ac)^3}}. \tag{12, 13}$$

The second derivatives are shown in Figure 7.

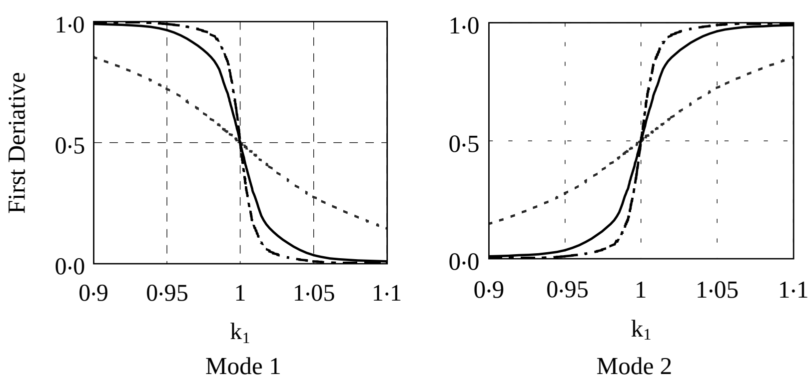

Figure 6. First derivatives of eigenvalues versus k_1 ($k_2 = 1$, $m_1 = m_2 = 1$). ····· $k = 0{\cdot}05$, —— $k = 0{\cdot}01$, - - - - $k = 0{\cdot}005$.

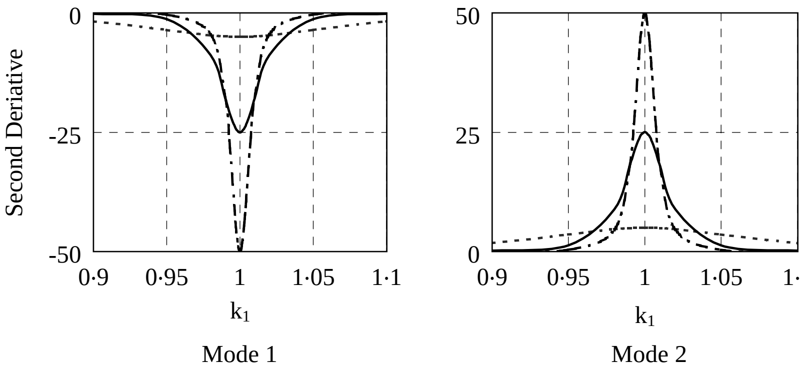

Figure 7. Second derivatives of eigenvalues versus k_1 ($k_2 = 1$, $m_1 = m_2 = 1$). ····· $k = 0{\cdot}05$, —— $k = 0{\cdot}01$, - - - - $k = 0{\cdot}005$.

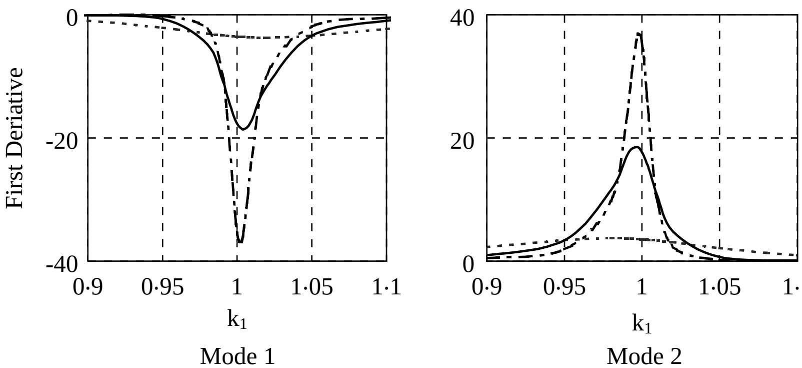

Figure 8. First derivatives of φ_1 versus k_1 ($k_2 = 1$, $m_1 = m_2 = 1$). ····· $k = 0{\cdot}05$, —— $k = 0{\cdot}01$, - - - - $k = 0{\cdot}005$.

The first derivatives of the eigenvectors with respect to k_1 are

$$\begin{Bmatrix} \dfrac{\partial \varphi_1}{\partial k_1} \\ \dfrac{\partial \varphi_2}{\partial k_1} \end{Bmatrix}_i = \begin{Bmatrix} -\dfrac{m_2 s_{\mathrm{i}}}{\sqrt{(m_1 + m_2 s_{\mathrm{i}}^2)^3}} \dfrac{\partial s_i}{\partial k_1} \\ \left[\dfrac{1}{\sqrt{m_1 + m_2 s_{\mathrm{i}}^2}} - \dfrac{m_2 s_{\mathrm{i}}^2}{\sqrt{(m_1 + m_2 s_{\mathrm{i}}^2)^3}} \right] \dfrac{\partial s_i}{\partial k_1} \end{Bmatrix}, \quad i = 1, 2, \tag{14}$$

where

$$\frac{\partial s_i}{\partial k_1} = \frac{1 - m_1(\partial \lambda_i / \partial k_1)}{k}. \tag{15}$$

Equation (15) implies that the derivative of eigenvector depends on the derivative of eigenvalue. The derivative of the eigenvector φ_1 versus the stiffness k_1 is illustrated in Figure 8.

From Figures 7 and 8, it is found that the second derivatives of the eigenvalues and the first derivatives of the eigenvectors with respect to the stiffness k_1 are in a similar form.

When $k = 0.05$, the derivatives show a smooth variation. At $k = 0{\cdot}05$, $k_1 = k_2 = 1$, and $m_1 = m_2 = 1$, the derivatives of the two modes are

$$\frac{\partial \lambda_{\rm i}}{\partial k_1} = \{0.5,\, 0{\cdot}5\}, \quad \frac{\partial^2 \lambda_{\rm i}}{\partial k_1^2} = \{-5,\, 5\},$$

$$\frac{\partial \boldsymbol{\varphi}_1}{\partial k_1} = \{-3{\cdot}53553, \quad 3{\cdot}53553\}^{\rm T}, \quad \frac{\partial \boldsymbol{\varphi}_2}{\partial k_1} = \{3.53553, \quad 3.53553\}^{\rm T}.$$

With the decrease of the stiffness k, the values of the derivatives increase greatly for k_1 approaching to 1. A narrow and violent curve comes from a very small k. When k is smaller than 0·01, a tiny decrease of k brings a large increase on the peak values, suggesting that the derivatives are strongly non-linear to k.

The slight difference between the derivatives of the eigenvalues and that of the eigenvectors is the location of the peak values. The second derivatives of the eigenvalues in equations (12) and (13) reach peak values only when

$$\frac{(k_1 - k)m_2}{(k_2 - k)m_1} = 1. \tag{16}$$

If $m_1/m_2 = 1$, the peak values must be at $k_1/{\rm k}_2 = 1$, so that it is at $k_1 = 1$ as in Figure 7.

The peak values for the derivatives of the eigenvectors appear at $k_1 \approx 1$, but never at $k_1 = 1$.

When k_1 is far from 1, the values of the second derivatives of the eigenvalues and the first derivatives of the eigenvectors are getting small as the stiffness k decreases. This means that the eigenvalues are veering less and the modes are not violently localized in the range of k_1 far from 1, even though k is small and the eigenvalues are close.

3.2. DERIVATIES WITH RESPECT TO MASS m_1

The derivatives of the eigenvalues with respect to the mass m_1 are computed as

$$\frac{\partial \lambda_1}{\partial m_1} = -\frac{\lambda_1}{m_1} + \frac{1}{2a}\left[k_2 + k + \frac{(k_2 + k)b + 2m_2 c}{\sqrt{b^2 - 4ac}}\right], \tag{17}$$

$$\frac{\partial \lambda_2}{\partial m_1} = -\frac{\lambda_2}{m_1} + \frac{1}{2a}\left[k_2 + k - \frac{(k_2 + k)b + 2m_2 c}{\sqrt{b^2 - 4ac}}\right], \tag{18}$$

$$\frac{\partial^2 \lambda_1}{\partial m_1^2} = -\frac{2}{m_1}\frac{\partial \lambda_1}{\partial m_1} - \frac{1}{2a}\left\{\frac{(k_2 + k)^2}{\sqrt{b^2 - 4ac}} - \frac{[(k_2 + k)b + 2m_2 c]^2}{\sqrt{(b^2 - 4ac)^3}}\right\}, \tag{19}$$

$$\frac{\partial^2 \lambda_2}{\partial m_1^2} = -\frac{2}{m_1}\frac{\partial \lambda_2}{\partial m_1} + \frac{1}{2a}\left\{\frac{(k_2 + k)^2}{\sqrt{b^2 - 4ac}} - \frac{[(k_2 + k)b + 2m_2 c]^2}{\sqrt{(b^2 - 4ac)^3}}\right\}. \tag{20}$$

The second derivatives of the eigenvalues calculated from equations (19) and (20) are given in Figure 9.

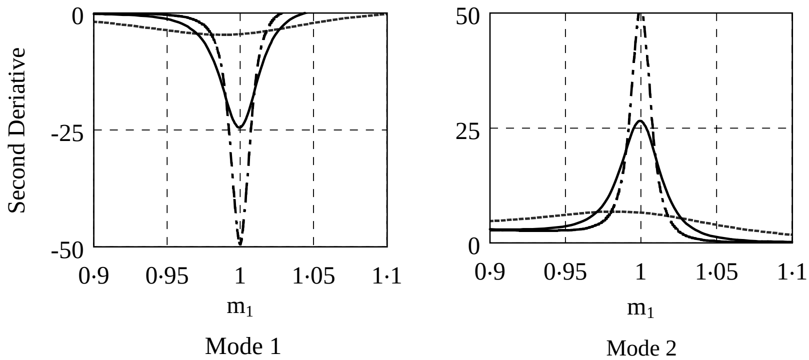

Figure 9. Second derivatives of eigenvalues versus m_1 ($k_1 = k_2 = 1$, $m_2 = 1$). ····· $k = 0{\cdot}05$, —— $k = 0{\cdot}01$, - - - - $k = 0{\cdot}005$.

The first derivatives of the eigenvectors with respect to m_1 are

$$\begin{Bmatrix} \dfrac{\partial \varphi_1}{\partial m_1} \\ \dfrac{\partial \varphi_2}{\partial m_1} \end{Bmatrix}_i = \begin{Bmatrix} -\dfrac{1 + 2m_2 s_i \dfrac{\partial s_i}{\partial m_1}}{2\sqrt{(m_1 + m_2 s_i^2)^3}} \\ \dfrac{\dfrac{\partial s_i}{\partial m_1}}{\sqrt{m_1 + m_2 s_i^2}} - \dfrac{\left(1 + 2m_2 \dfrac{\partial s_i}{\partial m_1}\right) s_i}{2\sqrt{(m_1 + m_2 s_i^2)^3}} \end{Bmatrix}, \; i = 1, 2, \tag{21}$$

where

$$\frac{\partial s_i}{\partial m_1} = -\frac{\lambda_i + m_1(\partial \lambda_i / \partial m_1)}{k}. \tag{22}$$

The derivatives of φ_1 are shown in Figure 10.

The derivatives with respect to the mass resemble that to the stiffness. For $k = 0{\cdot}05$, $k_1 = k_2 = 1$, and $m_1 = m_2 = 1$, the derivatives are

$$\frac{\partial \lambda_i}{\partial m_1} = \{-0{\cdot}5, -0.55\}, \; \frac{\partial^2 \lambda_i}{\partial m_1^2} = \left\{-4{\cdot}5, \quad 6{\cdot}6\right\},$$

$$\frac{\partial \boldsymbol{\varphi}_1}{\partial m_1} = \{3{\cdot}35876, -3.71231\}^{\mathrm{T}}, \; \frac{\partial \boldsymbol{\varphi}_2}{\partial m_1} = \{-4{\cdot}01197, -3{\cdot}79805\}^{\mathrm{T}}.$$

There are some differences between these values and those with respect to k_1. The values are not same for two modes. The peak values of the second derivatives of the eigenvalues are no longer at $k_1 = 1$.

By comparison of the figures with the figures in section 2, it is noted that the second derivatives of the eigenvalues and the first derivatives of the eigenvectors match the trend of the veering and localization. The values of the derivatives clearly show whether or not the phenomena are violent. Consequently, the derivatives are suggested to be the numerical measures to distinguish the occurrence and the degree of the phenomena.

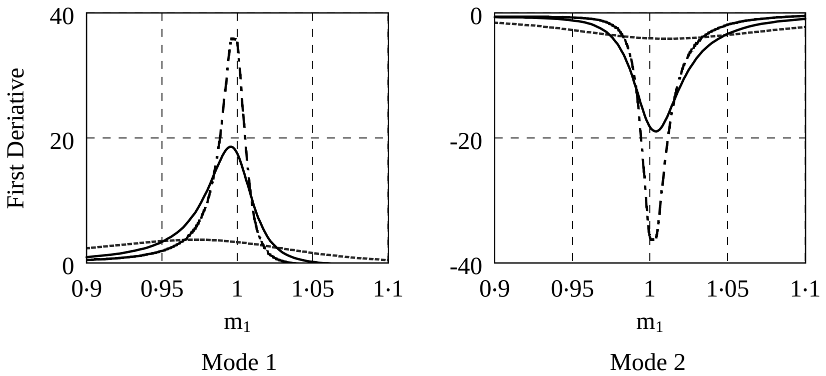

Figure 10. First derivatives of φ_1 versus m_1 ($k_1 = k_2 = 1, m_2 = 1$). ····· $k = 0{\cdot}05$, —— $k = 0{\cdot}01$, - - - - $k = 0{\cdot}005$.

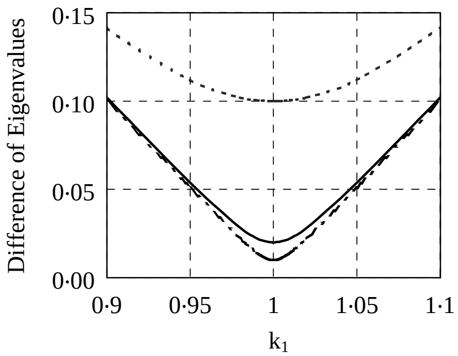

Figure 11. Difference of two eigenvalues versus k_1 ($k_2 = 1$, $m_1 = m_2 = 1$). ····· $k = 0{\cdot}05$, —— $k = 0{\cdot}01$, - - - - $k = 0{\cdot}005$.

4. RELATION OF DERIVATIVES TO CLOSE EIGENVALUES

To define if two eigenvalues are close must be based on their effect. Therefore, whether or not two eigenvalues are close is measured by taking the numerical measure of the curve veering or mode localization into consideration.

4.1. DERIVATIVES AND EIGENVALUE DIFFERENCE

The difference of two eigenvalues can be obtained from equations (3) and (4) as

$$\Delta\lambda = \lambda_2 - \lambda_1 = \frac{\sqrt{b^2 - 4ac}}{a}. \tag{23}$$

The difference versus k_1 is shown in Figure 11. The smallest difference of two eigenvalue appears at $k_1 = 1$, for a given k.

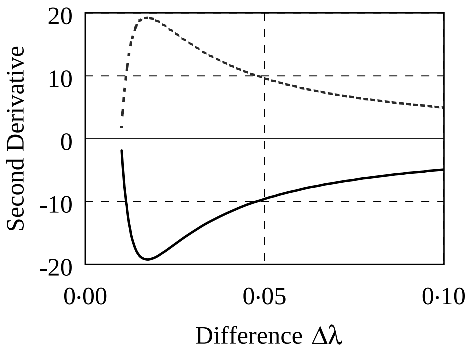

Figure 12. Second derivatives of eigenvalues versus eigenvalue difference ($k_1 = 0{\cdot}99$, $k_2 = 1$, k varies, $m_1 = m_2 = 1$). —— Mode 1, ····· Mode 2.

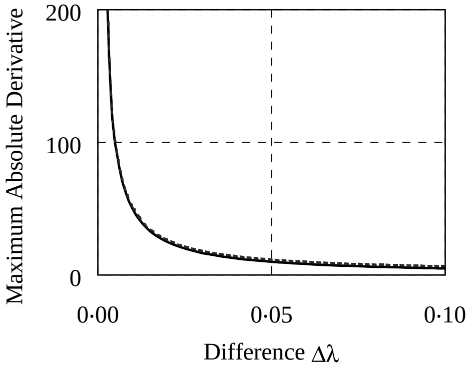

Figure 13. Absolute second derivatives of eigenvalues versus eigenvalue difference ($k_1 = k_2 = 1$, k varies, $m_1 = m_2 = 1$). —— $\dfrac{\partial^2 \lambda}{\partial k_1^2}$, ····· $\dfrac{\partial^2 \lambda}{\partial m_1^2}$.

Equations (12) and (13) can be expressed as

$$\frac{\partial^2 \lambda_1}{\partial k_1^2} = -\frac{2k^2}{m_1^3 m_2 \Delta\lambda^3}, \qquad \frac{\partial^2 \lambda_2}{\partial k_1^2} = \frac{2k^2}{m_1^3 m_2 \Delta\lambda^3}, \tag{24, 25}$$

which are rewritten as

$$\frac{\partial^2 \lambda_1}{\partial k_1^2} = -\frac{1}{2m_1^2 \Delta\lambda} + \frac{[(k_1 + k)m_2 - (k_2 + k)m_1]^2}{2m_1^4 m_2^2 \Delta\lambda^3}, \tag{26}$$

$$\frac{\partial^2 \lambda_2}{\partial k_1^2} = \frac{1}{2m_1^2 \Delta\lambda} - \frac{[(k_1 + k)m_2 - (k_2 + k)m_1]^2}{2m_1^4 m_2^2 \Delta\lambda^3}. \tag{27}$$

If k_1 is given to be 0·99, the relation between the derivatives and the difference is shown in Figure 12. The variation of the difference only comes from the stiffness k. It is noted that a small difference is not always a guarantee for an increase on the value of the derivatives.

Considering the special case with $k_1 = k_2$ and $m_1 = m_2$, the second terms in equations (26) and (27) become zero. The eigenvalue difference and the second derivatives can be

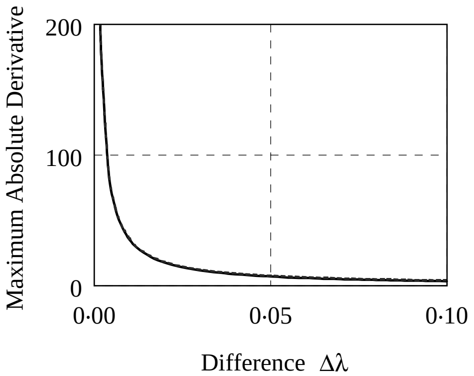

Figure 14. Absolute first derivatives of eigenvectors versus eigenvalue difference ($k_1 = k_2 = 1$, k varies, $m_1 = m_2 = 1$). —— $\dfrac{\partial^2 \boldsymbol{\varphi}}{\partial k_1}$, ····· $\dfrac{\partial^2 \boldsymbol{\varphi}}{\partial m_1}$.

reduced to

$$\Delta\lambda = \frac{2k}{m_1}, \qquad \frac{\partial^2 \lambda_1}{\partial k_1^2} = -\frac{1}{2m_1^2 \Delta\lambda} = -\frac{1}{4m_1 k}, \tag{28, 29}$$

$$\frac{\partial^2 \lambda_2}{\partial k_1^2} = \frac{1}{2m_1^2 \Delta\lambda} = \frac{1}{4m_1 k}. \tag{30}$$

The equations suggests that $\Delta\lambda$ is linear to k and the second derivatives of the eigenvalues are reciprocal to k or to $\Delta\lambda$. However, they are all independent from the spring stiffness k_1 and k_2. Given $m_1 = 1$, the eigenvalue difference only depends on the stiffness k. So do the derivatives. The second derivatives of the eigenvalues computed from equations (29) and (30) are plotted in Figure 13. When $\Delta\lambda$ is 0·5, for which $k = 0{\cdot}25$, the value of the derivatives is 1. Therefore, the value of derivatives increases greatly for $\Delta\lambda$ smaller than 0·5 or k smaller than 0·25.

It is seen that only in the case, in which $k_1 = k_2$ and $m_1 = m_2$, the derivatives increase from a small eigenvalue difference, which means that the eigenvalues are much close.

Similar relation between the first derivatives of the eigenvectors with respect to k_1 and the eigenvalue difference holds, as shown in Figure 14. It is known that the maximum values are not at $k_1 = k_2 = 1$ and $m_1 = m_2 = 1$.

The largest absolute derivatives with respect to m_1 at $k_1 = k_2 = 1$ and $m_1 = m_2 = 1$ are also given in Figures 13 and 14. There is a little difference between the values and those with respect to k_1.

4.2. CRITICAL VALUES

A value of the derivative of eigenvalue or eigenvector can be given as a mark to distinguish the occurrence of the veering or localization. Those greater than the value suggest a violent veering and localization. Once this critical value is given, to check the occurrence and the degree of the phenomena becomes easy. It is also easy to find if two close eigenvalues can lead to the veering and localization.

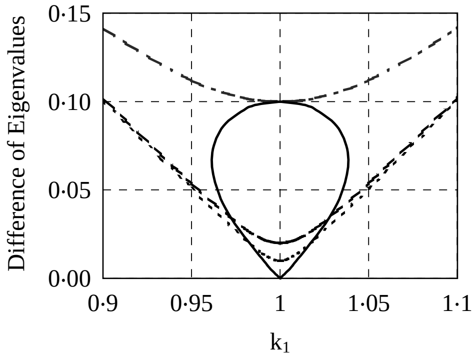

Figure 15. Area with absolute second derivatives of eigenvalues equal to or greater than 5 ($k_2 = 1, m_1 = m_2 = 1$). ---- $k = 0{\cdot}05$, ---- $k = 0{\cdot}01$, ····· $k = 0{\cdot}005$, ——— $\dfrac{\partial^2 \lambda}{\partial k_1^2} = \pm 5$.

In Figure 15, a curve, on which the second derivatives of the eigenvalues with respect to k_1 are equal to 5, is drawn. The derivatives within the heart-shaped area, surrounded by the curve, should be greater than 5. Out of this area, the derivatives are smaller than 5, though an eigenvalue difference can be small. On this area, a composition of k_1 and k contributes the second derivative to be equal to or greater than 5. The area locates at $0{\cdot}96151 < k_1 < 1{\cdot}03849$ and $k < 0{\cdot}05$. In other words, a derivative is guaranteed to be smaller than 5 if $k_1 < 0{\cdot}96151$, or $k_1 > 1{\cdot}03849$, or $k > 0{\cdot}05$.

If the critical value is given as 5, a derivative equal to or great than 5 is considered the existence of the veering and the eigenvalue difference small enough for the eigenvalues to be close. Therefore, any k_1 and k located within the area can cause the eigenvalues to be close and the veering to occur. If the critical value is taken to be smaller than 5, the heart-shaped area is larger than that in Figure 15. Otherwise, the area is smaller.

The area of the first derivatives of the eigenvectors with respect to k_1 equal to or greater than 5 is given in Figure 16. The area is in $0{\cdot}95357 < k_1 < 1{\cdot}04643$.

Certainly, similar heart-shaped areas can be obtained for the derivatives with respect to m_1.

5. CONCLUSION

The derivatives of the eigenvalues and eigenvectors have been examined for the phenomena of the curve veering and mode localization. It is demonstrated that the second derivative of the eigenvalue and the first derivative of the eigenvector can be taken as the numerical measures for the occurrence and the degree of the phenomena. The occurrence can be checked through a comparison of the derivatives with a critical value. This value should depend on a specific dynamic problem and must be given by taking the engineering orientation into consideration.

The definition of close eigenvalues is based on their effect on the veering or localization. Two eigenvalues are considered to be close if their existence brings the occurrence of the phenomena. From this point of view, two approaching eigenvalues are not always close, though their difference can be very small.

The analysis is shown in weakly coupled springs. The weak coupling is the dominance for the eigenvalues to be close. A tiny change of the coupling spring likely causes a violent veering and localization, which are numerically presented by the second derivatives of the

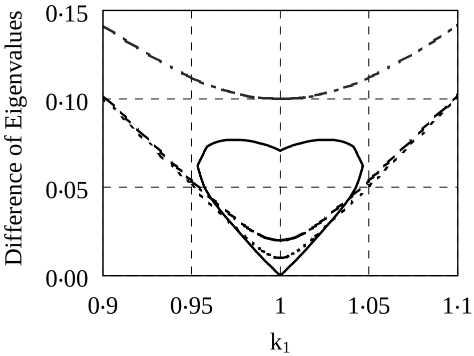

Figure 16. Area with absolute first derivatives of eigenvectors equal to or greater than 5 ($k_2 = 1$, $m_1 = m_2 = 1$). ---- $k = 0{\cdot}05$, --- $k = 0{\cdot}01$, ····· $k = 0{\cdot}005$, —— $\dfrac{\partial^2 \boldsymbol{\varphi}}{\partial k_1^2} = \pm 5$.

eigenvalues and the first derivatives of the eigenvectors. A heart-shaped area defines the two eigenvalues to be close, on which the phenomenon of veering or localization exists.

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REFERENCES

1. P. W. ANDERSON 1958 *Physics Review* **109**, 1492–1505. Absence of diffusion in certain random lattices.
2. C. H. HODGES 1982 *Journal of Sound and Vibration* **82**, 441–424. Confinement of vibration by structural irregularity.
3. O. O. BENDIKSEN 1987 *American Institute of Aeronautics and Astronautics Journal* **25**, 1241–1248. Mode localization phenomena in large space structures.
4. C. PIERRE and E. H. DOWELL 1987 *Journal of Sound and Vibration* **114**, 547–564. Localization of vibrations by structural irregularity.
5. D. AFOLABI 1988 *Journal of Sound and Vibration* **122**, 535–545. A note on the rouge failure of turbine blades.
6. S. T. WEI and C. PIERRE 1988 *Journal of Vibration, Acoustics, Stress, and Reliability in Design* **110**, 429–439. Localization phenomena in mistuned assemblies with cyclic symmetry, part I: free vibration.
7. P. J. CORNWELL and O. O. BENDIKSEN 1989 *American Institute of Aeronautics and Astronautics Journal* **27**, 219–226. Localization of vibrations in large space reflectors.
8. A. F. VAKAKIS 1992 *Journal of Sound and Vibration* **158**, 341–361. Nonsimilar normal oscillations in a strongly nonlinear discrete system.
9. A. F. VAKAKIS, T. NAYFEH and M. E. KING 1993 *Journal of Applied Mechanics* **60**, 388–397. A multiple-scales analysis of nonlinear, localized modes in a cyclic periodic system.
10. S. W. SHAW and C. PIERRE 1994 *Journal of Sound and Vibration* **169**, 319–347. Normal modes of vibration for nonlinear continuous systems.
11. M. E. KING and A. F. VAKAKIS 1994 *American Institute of Aeronautics and Astronautics Journal* **116**, 332–340. An energy-based formulation for computing nonlinear normal modes in undamped continuous system.

12. M. Petyt and C. C. Fleischer 1971 *Journal of Sound and Vibration* **18**, 17–30. Free vibration of a curved beam.
13. A. W. Leissa 1974 *Journal of Applied Mathematics and Physics* (ZAMP) **25**, 99–111. On a curve veering aberration.
14. J. R. Kutter and V. G. Sigillito 1981 *Journal of Sound and Vibration* **75**, 585–588. On curve veering.
15. M. S. Triantafyllou 1984 *The Shock and Vibration Digest* **16**, 9–17. Linear dynamics of cables and chains.
16. G. S. Schajer 1984 *Journal of Sound and Vibration* **92**, 11–19. The vibration of a rotating circular string subject to a fixed end restraint.
17. N. C. Perkins and C. D. Mote Jr 1986 *Journal of Sound and Vibration* **106**, 451–463. Comments on curve veering in eigenvalue problems.
18. C. Pierre, D. M. Tang and E. H. Dowell 1987 *American Institute of Aeronautics and Astronautics Journal* **25**, 1249–1257. Localized vibrations of disordered multi-span beams: theory and experiment.
19. C. Pierre 1988 *Journal of Sound and Vibration* **126**, 485–502. Mode localization and eigenvalue loci veering phenomena in disordered structures.